University of California, Berkeley
Physics 110A Spring 2003 Section 2 (*Strovink*)

Solution Set 3

1. The basic facts that we will use here are that the electric field inside each plate must vanish, and that each charge density produces a constant electric field proportional to it which is positive above it and negative below it. So, consider first the region inside the first plate, at the bottom. The electric field gets a positive contribution from Q_1^b below it and a negative contribution from Q_1^t, Q_2 and Q_3, so we need

$$Q_1^b - Q_1^t - Q_2 - Q_3 = 0. \tag{1}$$

Of course, we also have that $Q_1 = Q_1^b + Q_1^t$, and combining these equations, we see that,

$$Q_1^b = \frac{1}{2}\left(Q_1 + Q_2 + Q_3\right) \tag{2}$$

$$Q_1^t = \frac{1}{2}\left(Q_1 - Q_2 - Q_3\right). \tag{3}$$

Similarly, we find that for the second and third plates,

$$Q_2^b - Q_2^t + Q_1 - Q_3 = 0, \tag{4}$$

$$Q_3^b - Q_3^t + Q_1 + Q_2 = 0, \tag{5}$$

$$Q_2^b + Q_2^t = Q_2, \tag{6}$$

$$Q_3^b + Q_3^t = Q_3, \tag{7}$$

leading finally to the expressions,

$$Q_1^b = \frac{1}{2}\left(Q_1 + Q_2 + Q_3\right) \tag{8}$$

$$Q_1^t = \frac{1}{2}\left(Q_1 - Q_2 - Q_3\right) \tag{9}$$

$$Q_2^b = \frac{1}{2}\left(-Q_1 + Q_2 + Q_3\right) \tag{10}$$

$$Q_2^t = \frac{1}{2}\left(Q_1 + Q_2 - Q_3\right) \tag{11}$$

$$Q_3^b = \frac{1}{2}\left(-Q_1 - Q_2 + Q_3\right) \tag{12}$$

$$Q_3^t = \frac{1}{2}\left(Q_1 + Q_2 + Q_3\right). \tag{13}$$

2. (a) Taking an electron to be a uniformly charged sphere of radius r_0 (so its charge density is $\rho_0 = \frac{e}{\frac{4}{3}\pi r_0^3}$), we can compute its electrostatic self energy by computing,

$$U = \frac{\epsilon_0}{2}\int E^2 d\tau. \tag{14}$$

We use Gauss's Law to compute $\vec{E}$. First, consider a spherical Gaussian surface inside the electron. This gives us,

$$4\pi r^2 E(r) = \int_0^{2\pi}\int_0^{\pi}\int_0^{r}\frac{\rho_0}{\epsilon_0}r'^2 dr' \sin\theta' d\theta' d\phi' = \frac{4\rho_0\pi r^3}{3\epsilon_0} \tag{15}$$

$$\Rightarrow E(r < r_0) = \frac{\rho_0 r}{3\epsilon_0}, \tag{16}$$

for the field inside the sphere, and for a Gaussian surface outside we find,

$$4\pi r^2 E(r) = \int_0^{2\pi}\int_0^{\pi}\int_0^{r_0}\frac{\rho_0}{\epsilon_0}r'^2 dr' \sin\theta' d\theta' d\phi' = \frac{4\rho_0\pi r_0^3}{3\epsilon_0} \tag{17}$$

$$\Rightarrow E(r > r_0) = \frac{\rho_0 r_0^3}{3\epsilon_0 r^2}, \tag{18}$$

Thus, we find that,

$$U = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{2\pi\rho_0^2}{9\epsilon_0}\left(\int_0^{r_0} r^4 dr + \int_{r_0}^{\infty}\frac{r_0^6}{r^2}dr\right) = \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 r_0}. \tag{19}$$

Now, suppose that this self energy is equal to its mass energy,

$$U = m_e c^2 = \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 r_0}, \tag{20}$$

then we see that,

$$r_0 = \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 m_e c^2} = \frac{3}{5}\frac{(1.6\times 10^{-19})^2 C^2 \times 9.0\times 10^9 V\cdot m/C}{5.1\times 10^5 eV \times 1.6\times 10^{-19} C\cdot V/eV} = 1.7\times 10^{-15} m = 1.7 fm \tag{21}$$

(b) Using Gauss's Law, we see that as the charge inside the shell vanishes, (and as one can also see by symmetry) the electric field there vanishes. Outside the sphere, the electric field is just that of a point charge, so we see that,

$$U = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{4\pi\epsilon_0}{2}\int_{r_0}^{\infty}\left(\frac{e}{4\pi\epsilon_0 r^2}\right)^2 r^2 dr = \frac{e^2}{8\pi\epsilon_0 r_0}. \tag{22}$$

Equating this with the rest mass of the electron, we find that

$$r_0 = \frac{1}{2}\frac{e^2}{4\pi\epsilon_0 m_e c^2} = 1.4 fm \tag{23}$$

3. As the size of each of the sheets is far more than their thickness, we can think of this configuration as consisting of n parallel plate capacitors in parallel, with a total capacitance given by

$$nC = nA\epsilon/d = 2.3n(1m^2)\epsilon_0/(2.5\times 10^{-5}m) = 8.1n\times 10^{-7} As/V = 2.3n\times 10^{-10} Ah/V. \tag{24}$$

We apply $250V$ to this configuration, and we want

$$Q = nCV = 500Ah = 250V \times 2.3n\times 10^{-10} Ah/V \Rightarrow n = 8.8\times 10^9. \tag{25}$$

The height of this configuration is,

$$(2n+1)\times 2.5\times 10^{-5}m = 4.4\times 10^5 m. \tag{26}$$

4. We use the method of images, noting that this configuration is equivalent to a pair of wires, one over the conductor at a distance D, and its mirror, underneath it a distance D with the opposite charge. Thus, we expect that the force per unit length exerted on the upper wire should be equivalent to the force per unit length exerted by the mirror wire. We can easily compute this by noting that this is just the product of the electric field produced by the mirror wire evaluated at the original wire multiplied by λ, its charge per unit length. First, we compute the electric field of an infinite wire of charge per unit length $-\lambda$ using Gauss's law on a cylindrical Gaussian surface C of length l using coordinates where the wire is extended along the z-axis,

$$\begin{aligned}\int_C \vec{E}\cdot da = 2\pi r l E(r) &= \frac{1}{\epsilon_0}\int -\lambda\delta(x)\delta(y)d\tau = -\frac{\lambda l}{\epsilon_0}\\ \Rightarrow E(r) &= -\frac{\lambda}{2\pi r\epsilon_0}.\end{aligned} \tag{27}$$

As the wires are a length $2D$ apart, we see that the force per unit length exerted on the wire by the conducting plate is just,

$$F = \lambda E(2D) = -\frac{\lambda^2}{4\pi\epsilon_0 D}. \tag{28}$$

By Newton's second Law, this is equal and opposite to the force we wished to compute. For fun, we'll also compute this force another way, by first finding the force per unit area on the conductor due to the wire and then integrating over y, thereby showing the validity of Newton's second law in this configuration. Now, to do this, we need to compute the induced surface charge on the conductor, which is related to the discontinuity in the electric field at the surface of the conductor. The wire is extended along the x direction and is centered at $z = D$ and $y = 0$, rather than along the z direction at $x = y = 0$ as we are used to. We simply inorporate this by just re-defining $r = \sqrt{y^2 + (z \pm D)^2}$ (the $\pm$ is $-$ for the actual wire and $+$ for its image) and use the above formula for the electric field,

$$E(y,z) = \mp \frac{\lambda}{2\pi\epsilon_0 \sqrt{y^2 + (z \pm D)^2}}. \tag{29}$$

However, we must now remember that the field is directed radially along the *new* r direction, which means that in components we have,

$$E_y(y,z) = \frac{y}{\sqrt{y^2 + (z \pm D)^2}} E(y,z) \tag{30}$$

$$E_z(y,z) = \frac{(z \pm D)}{\sqrt{y^2 + (z \pm D)^2}} E(y,z) \tag{31}$$

As we are only concerned with the electric field at $z = 0$, notice that by symmetry the electric fields of the wire and its image cancel in the y direction but add in the z direction (as must also be the case for the electric field along the conducting surface to vanish), and so the electric field at the surface of the conductor is just twice the z-component of the field produced there by a single wire,

$$E_z(y, z = 0) = 2 \times \frac{-D\lambda}{2\pi\epsilon_0 (y^2 + D^2)} \tag{32}$$

The induced surface charge is just, using Griffiths (2.48),

$$\sigma = \epsilon_0 E_z = -\frac{D\lambda}{\pi(y^2 + D^2)}, \tag{33}$$

and the force per unit area exerted on an element of the surface charge by the wire is directed outward normally with a magnitude given by,

$$f = \frac{1}{2\epsilon_0}\sigma^2 = \frac{D^2\lambda^2}{2\pi^2\epsilon_0 (D^2 + y^2)^2}. \tag{34}$$

where the factor of 1/2 is because we don't want to include the effect of the electric field produced by the image charges acting on the conductor! If we now integrate this result over all y, we find that (using the substitution $y = D\tan\theta$),

$$F = \frac{D^2\lambda^2}{2\pi^2\epsilon_0} \int_{-\infty}^{\infty} \frac{dy}{(y^2 + D^2)^2} = \frac{\lambda^2}{2\pi^2\epsilon_0 D} \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = \frac{\lambda^2}{4\pi\epsilon_0 D}. \tag{35}$$

5.

6. (a) In order to have the component of the electric field along the conductor vanish, we see that the image charge must have the opposite sign and be at $(0, 0, -\epsilon)$. Thus, the electric field at $(0, 0, u)$ is just the sum of the fields due to both these point charges,

$$\mathcal{E} = \frac{q}{4\pi\epsilon_0 (u - \epsilon)^2} - \frac{q}{4\pi\epsilon_0 (u + \epsilon)^2} \tag{36}$$

Now, using the first order Taylor expansion,

$$(1 \pm x)^{-2} = 1 \mp 2x + \ldots, \tag{37}$$

we see that the leading dependence on u is given by,

$$\mathcal{E} \approx \frac{q}{4\pi\epsilon_0 u^2}\left(1 + 2\frac{\epsilon}{u} - (1 - 2\frac{\epsilon}{u})\right) = \frac{q\epsilon}{\pi\epsilon_0 u^3}. \tag{38}$$

(b) We'd like to find a set of image charges such that the potential is constant along the $z = 0$ and $y = 0$ planes. It is easy to guess such a choice - in order to have vanishing electric fields on both the intersecting planes, we need image charges in each quadrant, with opposite charges across each plane. Thus, we need a charge $-q$ at both $(0, -\epsilon/\sqrt{2}, \epsilon/\sqrt{2})$ and $(0, \epsilon/\sqrt{2}, -\epsilon/\sqrt{2})$, as well as a charge q at $(0, -\epsilon/\sqrt{2}, -\epsilon/\sqrt{2})$. To check that this is correct, we compute the potential at any point in the $z = 0$ and $y = 0$ planes,

$$\begin{aligned} V(x,y) =& \frac{q}{4\pi\epsilon_0 \sqrt{x^2 + (y - \epsilon/\sqrt{2})^2 + \epsilon^2/2}} - \frac{q}{4\pi\epsilon_0 \sqrt{x^2 + (y + \epsilon/\sqrt{2})^2 + \epsilon^2/2}} \\ &- \frac{q}{4\pi\epsilon_0 \sqrt{x^2 + (y - \epsilon/\sqrt{2})^2 + \epsilon^2/2}} + \frac{q}{4\pi\epsilon_0 \sqrt{x^2 + (y + \epsilon/\sqrt{2})^2 + \epsilon^2/2}} = 0, \end{aligned} \tag{39}$$

$$\begin{aligned} V(x,z) =& \frac{q}{4\pi\epsilon_0 \sqrt{x^2 + (z - \epsilon/\sqrt{2})^2 + \epsilon^2/2}} - \frac{q}{4\pi\epsilon_0 \sqrt{x^2 + (z - \epsilon/\sqrt{2})^2 + \epsilon^2/2}} \\ &- \frac{q}{4\pi\epsilon_0 \sqrt{x^2 + (z + \epsilon/\sqrt{2})^2 + \epsilon^2/2}} + \frac{q}{4\pi\epsilon_0 \sqrt{x^2 + (z + \epsilon/\sqrt{2})^2 + \epsilon^2/2}} = 0, \end{aligned} \tag{40}$$

and indeed they vanish. In order to compute the magnitude of the electric field at $(0, u/\sqrt{2}, u/\sqrt{2})$, it is convenient to rotate our coordinate system such that the original charge as well as the point we are considering are both along the z-axis. The four charges for the edges of a square in the zy plane, with charges $+q$ at $(0, 0, \pm\epsilon)$ and $-q$ at $(0, \pm\epsilon, 0)$. It is clear that the electric field points in the z-direction, and we can get the magnitude of the electric field by summing the z components of the electric fields due to each of the four point charges,

$$\mathcal{E} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{(u-\epsilon)^2} + \frac{1}{(u+\epsilon)^2} - 2\frac{u}{(u^2+\epsilon^2)^{3/2}} \right). \tag{41}$$

Using Taylor approximations to higher order,

$$(1+x)^m = 1 + mx + (m(m-1)/2)x^2 + \ldots, \tag{42}$$

we see that when $\epsilon \ll u$,

$$\begin{aligned} \mathcal{E} &\approx \frac{q}{4\pi\epsilon_0 u^2} \left(1 + 2\frac{\epsilon}{u} + 3\frac{\epsilon^2}{u^2} + 1 - 2\frac{\epsilon}{u} + 3\frac{\epsilon^2}{u^2} - 2\left(1 - (3/2)\frac{\epsilon^2}{u^2}\right)\right) \\ &= \frac{9q\epsilon^2}{4\pi\epsilon_0 u^4}. \end{aligned} \tag{43}$$

(c) The idea here is that the previous two cases were the cases $n = 1$ and $n = 2$, and we just need to extend the logic further. In particular, the basic method is as follows. Start with the original charge and construct images for it associated with both the planes (that is, charges of the opposite sign placed at the position of the reflection of the original charge across the plane). Now, both the images have one image charge (namely, the original charge), but might need images along the plane they were not associated with (in the $n = 2$ case, this was accomplished through a single image, which will not be the case for higher n). Now, these images of image charges may further need images along the first plane, and so on. As the angle between the two planes is an integral fraction of π, this process terminates just as it did for $n = 2$ when the final image charge added already has its images along both planes present. It is easy to see that this configuration will correspond to alternating charges arranged on the vertices of a regular plane polygon with $2n$ sides. Now, examining the pattern of the last two problems, we would guess that the electric field at $x = 0$ along the bisecting plane on which the original charge was located should go as $u^{-(n+2)}$,

$$\mathcal{E} \propto \frac{\epsilon^n}{4\pi\epsilon_0 u^{n+2}}. \tag{44}$$

Let's try to show that this is in fact true. The potential due to these $2n$ charges at a point a distance u along the bisecting plane can be written explictly,

$$V(u) = \frac{q}{4\pi\epsilon_0}\sum_{m=1}^{2n}\frac{(-1)^m}{\sqrt{(u-\epsilon\cos\pi m/n)^2+\epsilon^2\sin^2\pi m/n}} = \frac{q}{4\pi\epsilon_0}\sum_{m=1}^{2n}\frac{(-1)^m}{\sqrt{u^2-2u\epsilon\cos\pi m/n+\epsilon^2}} \tag{45}$$

Thus, we have (with $x=\frac{\epsilon}{u}$),

$$E(u) = -\frac{\partial V(u)}{\partial u} = \frac{q}{4\pi\epsilon_0}\sum_{m=1}^{2n}\frac{(-1)^m(u-\epsilon\cos\pi m/n)}{(u^2-2u\epsilon\cos\pi m/n+\epsilon^2)^{3/2}} = \frac{q}{4\pi\epsilon_0 u^2}\sum_{m=1}^{2n}\frac{(-1)^m(1-x\cos\pi m/n)}{(1-2x\cos\pi m/n+x^2)^{3/2}}. \tag{46}$$

We would like to know the expansion of the sum as a power series in x, and find the first non-zero term, i.e. we'd like to Taylor expand it about $x=0$. To do this, we need to know the p^{th} partial derivative of the sum with respect to x evaluated at $x=0$,

$$\frac{\partial^p}{\partial x^p}\left(\sum_{m=1}^{2n}\frac{(-1)^m(1+x\cos\pi m/n)}{(1-2x\cos\pi m/n+x^2)^{3/2}}\right)\Bigg|_{x=0}. \tag{47}$$

This doesn't look easy to compute. However, there are a few observations that will allow us to extract what we need from this. First, since we always evaluate the partial derivatives at $x=0$, the denominators of all these expressions always just become 1, and can be ignored. Second, the expressions in the numerator are always just polynomials in x and $\cos\pi m/n$, and when we set $x=0$, only the terms indedpendent of x matter. Further, it is easy to convince yourself that when you take the p^{th} derivative, the maximum power of $\cos\pi m/n$ that survives after we let $x=0$ is p. Thus, the p^{th} derivative evaluated at $x=0$ is just a sum of terms each of the form,

$$\sum_{m=1}^{2n}(-1)^m\cos^q\pi m/n = \sum_{m=1}^{2n}\cos m\pi\cos^q\pi m/n, \tag{48}$$

with $q\le p$. Now, for $q<n$ and q odd and n even or vice-versa, the contributions from the m^{th} term and the $(m+n)^{th}$ term exactly cancel. Suppose that $q<n$ and q and n are both even. Then, we can repeatedly use the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ to rewrite equation (48) as a sum of terms of the form (with $2r<q<n$ and defining $x=-e^{2r\pi i/n}$),

$$\begin{aligned}\sum_{m=1}^{2n}(-1)^m\cos 2r\pi m/n &= \frac{1}{2}\sum_{m=1}^{2n}\left((-e^{2r\pi i/n})^m+(-e^{-2r\pi i/n})^m\right) = \frac{1}{2}\sum_{m=1}^{2n}\left(x^m+x^{-m}\right)\\ &= \frac{1}{2}\left(\frac{x(x^{2n}-1)}{x-1}+\frac{(x^{-2n}-1)}{1-x}\right)=0,\end{aligned} \tag{49}$$

where we've used the expression for the sum of a finite geometric series as well as the fact that $x^{\pm 2n} = e^{\pm 4\pi i r}=1$. Similarly, if $q<n$ and q and n are both odd, then we need to consider terms of the form,

$$\sum_{m=1}^{2n}(-1)^m\cos\frac{2r\pi m}{n}\cos\frac{\pi}{n} = \frac{1}{2}\sum_{m=1}^{2n}\left((-e^{\frac{(2r+1)\pi i}{n}})^m+(-e^{\frac{(2r-1)\pi i}{n}})^m+(-e^{-\frac{(2r+1)\pi i}{n}})^m+(-e^{-\frac{(2r-1)\pi i}{n}})^m\right)=0 \tag{50}$$

in the same way. Therefore, we find that the p^{th} derivative of the sum we were considering vanishes for all $p<n$, and the first non-zero term will give us a contribution to the electric field of the form we had guessed,

$$\mathcal{E}\propto\frac{x^n}{4\pi\epsilon_0 u^2}=\frac{\epsilon^n}{4\pi\epsilon_0 u^{n+2}}. \tag{51}$$

7.

8. (a) This is a perfect situation to apply seperation of variables in Cartesian coordinates to solve Laplace's equation for the potential along the positive y axis. We look for solutions which are of the product form, $V(x, y, z) = X(x)Y(y)Z(z)$. In particular, when $y = 0$, we know that

$$V(x, y = 0, z) = V_0 \cos\frac{\pi x}{L} \cos\frac{\pi z}{L}, \tag{52}$$

which immediately implies $X(x) = \cos\frac{\pi x}{L}$, and $Z(z) = \cos\frac{\pi z}{L}$. Thus, Laplace's equation becomes,

$$\frac{\nabla^2 V}{V} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{2\pi^2}{L^2} = 0 \tag{53}$$

Thus, we see that we must have $Y(y) = V_0 e^{-\sqrt{2}\pi y/L}$ (where we've assumed that $V \to 0$ as $y \to \infty$) and so,

$$V(x, y, z) = V_0 e^{-\sqrt{2}\pi y/L} \cos\frac{\pi x}{L} \cos\frac{\pi z}{L}. \tag{54}$$

To calculate the electric field along the y-axis, note that $\vec{E} = -\nabla V$, and we see that as $\sin 0 = 0$, $E_x = E_z \propto \sin 0 = 0$ and $\vec{E}$ points along the y direction,

$$E_y(y) = -\frac{\partial V}{\partial y} = \frac{\sqrt{2}\pi V_0}{L} e^{-\sqrt{2}\pi y/L}. \tag{55}$$

Thus, the electric field decays exponentially in the y-direction with a characteristic length scale given by $\frac{L}{\sqrt{2}\pi}$.

(b) If this were a 2D problem, we the only change would be that the $Z(z)$ term would no longer be present in equation (53) above, thereby reducing by a factor of $\sqrt{2}$ the coefficient of the exponential, $V(y) = V_0 e^{-\pi y/L}$. This leads to a decay constant of $\frac{L}{\pi}$, which indicates a slower decay of the electric field in 2D.

(c) We can calculate the surface charge on the conductor by computing the discontinuity of the electric field there. Outside, the tubing, the electric field vanishes. As we approach $x = L/2$ from the inside, we see that the only non-zero component of the electric field is,

$$E_x(x = L/2, y > 0, z) = -\frac{\partial V}{\partial x} = \frac{\pi V_0}{L} e^{-\sqrt{2}\pi y/L} \cos\frac{\pi z}{L} \tag{56}$$

By using a tiny Guassian pillbox just spanning the surface of the tubing, we see that the surface charge density is proportional to this discontinuity,

$$\sigma(x = L/2, y > 0, z) = \epsilon_0(0 - E_x(x = L/2, y > 0, z)) = -\frac{\pi\epsilon_0 V_0}{L} e^{-\sqrt{2}\pi y/L} \cos\frac{\pi z}{L}. \tag{57}$$

Now, we can find the total induced charge on all four sides by just integrating this over the region $y > 0$ and $z \in [-L/2, L/2]$ (which is half of one of the four sides of the tubing) and multiplying by eight,

$$Q_i = 2 \times 4 \times \frac{-\pi\epsilon_0 V_0}{L} \int_0^\infty e^{-\sqrt{2}\pi y/L} dy \int_{-L/2}^{L/2} \cos\frac{\pi z}{L} dz = -8\sqrt{2}\epsilon_0 V_0. \tag{58}$$

The induced charge on the membrane can be computed as follows. Imagine making the conducting outer walls of the tubing thicker and consider a Gaussian surface within those walls. The electric field vanishes there and so must the total charge enclosed $Q_T = Q_m + Q_i = 0$, which gives us,

$$Q_m = -Q_i = 8\sqrt{2}\epsilon_0 V_0. \tag{59}$$